

Hidden Markov Model

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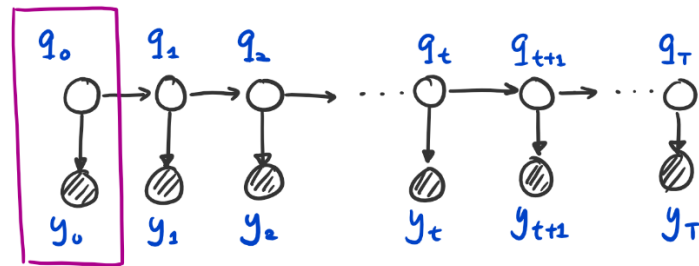
$$y = \{y_0, y_1, \dots, y_T\}$$

$$q = \{q_0, q_1, \dots, q_T\}$$

$$P(y) = \sum_q P(y, q)$$

$$= \sum_{q_0} \sum_{q_1} \dots \sum_{q_T} \underbrace{\pi(q_0)}_{\text{prior probability}} \prod_{t=0}^{T-1} \underbrace{P(q_{t+1}|q_t)}_{\text{transitional probability}} \prod_{t=0}^{T-1} \underbrace{P(y_t|q_t)}_{\text{emission probability}}$$

$$\pi(q_0)P(y_0|q_0)$$



$$P(q_t|y) = \frac{P(y|q_t)P(q_t)}{P(y)}$$

$$y = \{y_0, y_1, \dots, y_t, y_{t+1}, y_{t+2}, \dots, y_T\}$$

$$P(q_t|y) = \frac{P(y_0, y_1, \dots, y_t | q_t) P(q_t) P(y_{t+1}, \dots, y_T | q_t)}{P(y)}$$

$$= \frac{P(y_0, y_1, \dots, y_t, q_t) P(y_{t+1}, \dots, y_T | q_t)}{P(y)}$$

$$= \frac{\alpha(q_t) \beta(q_t)}{P(y)}$$

$$\begin{aligned}
\alpha(q_{t+1}) &= P(y_0, y_1 \dots y_t, y_{t+1}, q_{t+1}) \\
&= P(y_0, y_1 \dots y_t | q_{t+1}) P(y_{t+1} | q_{t+1}) P(q_{t+1}) \\
&= P(y_0 \dots y_t, q_{t+1}) P(y_{t+1} | q_{t+1}) \\
&= \sum_{q_t} P(y_0, y_1 \dots y_t, q_t, q_{t+1}) P(y_{t+1} | q_{t+1}) \\
&= \sum_{q_t} P(y_0, y_1 \dots y_t, q_{t+1} | q_t) P(q_t) P(y_{t+1} | q_{t+1}) \\
&= \sum_{q_t} \underbrace{P(y_0, y_1 \dots y_t | q_t)} P(q_{t+1} | q_t) \underbrace{P(q_t) P(y_{t+1} | q_{t+1})} \\
&= \sum_{q_t} P(y_0, y_1 \dots y_t, q_t) P(q_{t+1} | q_t) P(y_{t+1} | q_{t+1}) \\
&= \sum_{q_t} \alpha(q_t) \underbrace{a(q_t, q_{t+1})}_{\text{transitional probability}} \underbrace{b(q_{t+1})}_{\text{emission probability}}
\end{aligned}$$

$$\begin{aligned}
P(y) &= \sum_q P(y, q) \\
&= \sum_{q_0} \sum_{q_1} \dots \sum_{q_T} \pi(q_0) P(y_0 | q_0) \prod_{t=1}^T (P(q_{t+1} | q_t) P(y_t | q_t)) \\
&= \sum_{q_1} \sum_{q_2} \dots \sum_{q_T} \alpha(q_1) P(q_2 | q_1) P(y_1 | q_1) \prod_{t=2}^T (P(q_{t+1} | q_t) P(y_t | q_t)) \\
&= \sum_{q_2} \dots \sum_{q_T} \alpha(q_2) P(q_3 | q_2) P(y_2 | q_2) \prod_{t=3}^T (P(q_{t+1} | q_t) P(y_t | q_t)) \\
&= \sum_{q_T} \alpha(q_T)
\end{aligned}$$

$$\begin{aligned}
\beta(q_t) &= P(y_{t+1}, y_{t+2} \dots y_T | q_t) \\
&= \sum_{q_{t+1}} P(y_{t+1}, y_{t+2} \dots y_T, q_{t+1} | q_t) \\
&= \sum_{q_{t+1}} P(y_{t+1}, y_{t+2} \dots y_T | q_{t+1}, q_t) P(q_{t+1} | q_t) \\
&= \sum_{q_{t+1}} P(y_{t+1} | q_{t+1}) \underbrace{P(y_{t+2} \dots y_T | q_{t+1})}_{\beta(q_{t+1})} P(q_{t+1} | q_t) \\
&= \sum_{q_{t+1}} \beta(q_{t+1}) b(q_{t+1}) a(q_t, q_{t+1})
\end{aligned}$$

$$P(q_t | y) \triangleq r(q_t) = \frac{\alpha(q_t) \beta(q_t)}{P(y)}$$

$$r(q_t, q_{t+1}) = \frac{\alpha(q_t) a(q_t, q_{t+1}) b(q_{t+1}) \beta(q_{t+1})}{P(y)}$$

$$r(q_t) = \sum_{q_{t+1}} r(q_t, q_{t+1})$$

$$\hat{a}(q_t, q_{t+1}) = \frac{\sum_t r(q_t, q_{t+1})}{\sum_t r(q_t)}$$

$$\hat{b}(q_{t+1}) = \frac{\sum_t r(q_{t+1} | y_t = k)}{\sum_t r(q_t)}$$